# Infinite hierarchies of nonlocal symmetries of the Chen–Kontsevich–Schwarz type for the oriented associativity equations

**A Sergyeyev**

Mathematical Institute, Silesian University in Opava, Na Rybníčku 1, 746 01 Opava, Czech Republic

E-mail: Artur.Sergyeyev@math.slu.cz

## Abstract

We construct infinite hierarchies of nonlocal higher symmetries for the oriented associativity equations using solutions of associated vector and scalar spectral problems. The symmetries in question generalize those found by Chen, Kontsevich and Schwarz (*Nucl. Phys.* B **730** 352–63) for the WDVV equations. As a byproduct, we obtain a Darboux-type transformation and a (conditional) Bäcklund transformation for the oriented associativity equations.

PACS number: 02.30.Ik

## 1. Introduction

The Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations [1, 2], and the related geometric structures, in particular, the Frobenius manifolds [3–8], have attracted considerable attention because of their manifold applications in physics and mathematics.

More recently, the oriented associativity equations, a generalization of the WDVV equations, and the related geometric structures, *F*-manifolds, see e.g. [7–19], have also become a subject of intense research. These equations first appeared in [3, proposition 2.3] as the equations for the displacement vector. The oriented associativity equations describe *inter alia* isoassociative quantum deformations of commutative associative algebras [14, 15], cf also [16–19].

The oriented associativity equations (8) admit the gradient reduction (43) to the 'usual' associativity equations (44). Equations (44) and the so-called Hessian reduction (see [20–24] and [14]) of the oriented associativity equations naturally arise in topological 2D gravity [1, 25], singularity theory and complex geometry (see e.g. [8, 9]), and in differential geometry and theory of integrable systems, see [8], [26–36] and references therein.

There is a considerable body of work on the symmetry properties of the WDVV equations, see e.g. [37–41] for the point symmetries of the WDVV and generalized (in the sense of [42] and references therein) WDVV equations, [4, 43–45] and references therein for finite symmetries, Bäcklund transformations and dualities, and [26, 46–48] and references therein for the higher symmetries and (bi-)Hamiltonian structures for the WDVV equations, and also for (44), in three and four independent variables. Although the approach of [26, 46–48] in principle could [26] be generalized to the WDVV equations in more than four independent variables, this has not been done yet. Nevertheless, in [39, 40] infinite sets of *nonlocal* higher symmetries for the WDVV equations were found. To the best of our knowledge, higher (or generalized [49]) symmetries of the *oriented* associativity equations in arbitrary dimension were never fully explored.

The goal of the present paper is to construct nonlocal higher symmetries for the oriented associativity equations using, in analogy with [40], the solutions of auxiliary spectral problems. We show that the very solutions of the vector spectral problem (9), either *per se* or multiplied by a suitably chosen solution of the scalar spectral problem (10) with the opposite sign of the spectral parameter, indeed *are* (infinitesimal) symmetries for the oriented associativity equations (8); see theorem 1 below for details. The fact that solutions of (a) are symmetries for (8) is quite unusual in itself, as symmetries typically turn out to be quadratic [50] rather than linear in solutions of auxiliary linear problems.

Upon performing the gradient reduction to the 'usual' associativity equations (44) we reproduce the results of [40]; see corollaries 6 and 8. However, not all nonlocal symmetries from theorem 1 survive the gradient reduction and yield symmetries for (44), see corollary 6 and the subsequent discussion.

Expanding solutions of the spectral problems into formal Taylor series in the spectral parameter yields infinite hierarchies of nonlocal higher symmetries for (8) and (44); see corollaries 4 and 8.

Finally, as a byproduct, we obtain a Darboux-type transformation and some Bäcklund-type transformations relating the solutions of 'usual' and oriented associativity equations; see proposition 1 and corollaries 11 and 12 for further details. These transformations, as well as the nonlocal symmetries discussed above, could possibly yield new solutions for the oriented and 'usual' associativity equations.

## 2. Preliminaries

Let the Greek indices $\alpha, \beta, \gamma, \ldots$ (except for $\lambda, \mu, \eta, \zeta, \sigma, \tau, \chi, \phi, \psi$) run from 1 to $n$, where $n$ is a fixed natural number, and summation over the repeated indices be understood unless otherwise explicitly stated. In what follows we also assume that all functions under study are sufficiently smooth for all necessary derivatives to exist.

Consider the *oriented associativity equations*, see e.g. [10–14], for the structure 'constants' $c_{\alpha\beta}^{\delta}(x^1, \ldots, x^n)$ of a commutative $\left(c_{\nu\rho}^{\alpha} = c_{\rho\nu}^{\alpha}\right)$ algebra

$$c_{\alpha\rho}^{\nu} c_{\beta\gamma}^{\rho} = c_{\rho\gamma}^{\nu} c_{\alpha\beta}^{\rho}, \tag{1}$$

$$\frac{\partial c_{\beta\gamma}^{\alpha}}{\partial x^{\rho}} = \frac{\partial c_{\rho\gamma}^{\alpha}}{\partial x^{\beta}}. \tag{2}$$

Condition (1) means that the algebra in question is associative and (2) means that we consider isoassociative [14] quantum deformations of the algebra in question.

The oriented associativity equations (1), (2) can be written as compatibility conditions of the Gauss–Manin equations, see e.g. [4, 7, 14], for a scalar function $\chi(\lambda)$ (for the sake of brevity we shall often omit below the dependence on $x^1, \ldots, x^n$)

$$\frac{\partial^2 \chi(\lambda)}{\partial x^\alpha \partial x^\gamma} = \lambda c^\nu_{\alpha\gamma} \frac{\partial \chi(\lambda)}{\partial x^\nu}. \tag{3}$$

Here $\lambda$ is the spectral parameter. These equations have a very interesting interpretation, with $\chi$ playing the role of a wavefunction, in the context of quantum deformations of associative algebras [14].

We also have a zero-curvature representation for (1), (2) of the form (see e.g. [6, 42])

$$\frac{\partial \psi^\alpha(\lambda)}{\partial x^\beta} = \lambda c^\alpha_{\beta\gamma} \psi^\gamma(\lambda), \tag{4}$$

where we again omit, for the sake of brevity, the dependence of $\psi^\alpha$ on $x^1, \ldots, x^n$. In other words, equations (1), (2) are precisely the compatibility conditions for (4). The quantities $\psi^\alpha(\lambda)$ are nothing but the components of a generic vector field which is covariantly constant (in other terminology, parallel or flat) with respect to the covariant derivative associated with the one-parametric family of flat connections $-\lambda c^\alpha_{\nu\rho}$.

Upon introducing the quantities $\phi_\alpha(\lambda) = \partial \chi(\lambda)/\partial x^\alpha$ equation (3) can be written in the first-order form as

$$\partial \phi_\alpha(\lambda)/\partial x^\beta = \lambda c^\delta_{\alpha\beta} \phi_\delta(\lambda). \tag{5}$$

Quite obviously, the spectral problem (5) is, up to the change of sign of $\lambda$, adjoint to (4).

Now let $\chi^\alpha(\lambda), \alpha = 1, \ldots, n$, be the solutions of (3) normalized by the condition

$$\chi^\alpha(\lambda)|_{\lambda=0} = x^\alpha.$$

It is well known (see e.g. [4, 8, 14]) that $\chi^\alpha$ are nothing but flat coordinates for the one-parameter family $\lambda c^\alpha_{\nu\kappa}$ of flat connections (the flatness readily follows from (1) and (2)).

Following [3, 4], we can represent $\chi^\alpha$ in the form

$$\chi^\alpha(\lambda) = x^\alpha + \lambda K^\alpha + O(\lambda^2), \tag{6}$$

where $K^\alpha = K^\alpha(x^1, \ldots, x^n)$ is the so-called displacement vector. Plugging (6) into (3) and restricting our attention to the terms linear in $\lambda$ yields

$$c^\alpha_{\beta\gamma} = \frac{\partial^2 K^\alpha}{\partial x^\beta \partial x^\gamma}. \tag{7}$$

The ansatz (7) automatically solves (2), and (1) boils down to the overdetermined system

$$\frac{\partial^2 K^\nu}{\partial x^\alpha \partial x^\rho} \frac{\partial^2 K^\rho}{\partial x^\beta \partial x^\gamma} = \frac{\partial^2 K^\rho}{\partial x^\alpha \partial x^\beta} \frac{\partial^2 K^\nu}{\partial x^\rho \partial x^\gamma} \tag{8}$$

for $K^\alpha$. In what follows we shall refer to this system as the oriented associativity equations just as we referred to (1), (2), as, in combination with (7), equation (8) is equivalent to (1), (2) provided $c^\alpha_{\nu\rho} = c^\alpha_{\rho\nu}$.

Of course, the equations obtained by plugging (7) into (4), that is,

$$\frac{\partial \psi^\alpha(\lambda)}{\partial x^\beta} = \lambda \frac{\partial^2 K^\alpha}{\partial x^\beta \partial x^\gamma} \psi^\gamma(\lambda), \tag{9}$$

provide a zero-curvature representation for (8).

Likewise, plugging (7) into (3) yields a scalar spectral problem for (8)

$$\frac{\partial^2 \chi(\lambda)}{\partial x^\alpha \partial x^\gamma} = \lambda \frac{\partial^2 K^\nu}{\partial x^\alpha \partial x^\gamma} \frac{\partial \chi(\lambda)}{\partial x^\nu}. \tag{10}$$

## 3. Darboux-type transformation for oriented associativity equations

It is well known, see e.g. [4, appendix B], [37] and references therein, that there exist changes of variables that leave the second derivatives of the prepotential unchanged and map solutions of the (generalized) WDVV equations into new solutions. Quite interestingly, there is a change of variables of this kind that involves [37] a solution of the spectral problem (4), and thus can be thought of as a Darboux-type transformation.

It turns out that this transformation is readily generalized to the oriented associativity equations. Namely, the following assertion holds.

**Proposition 1.** *Let $K^\alpha$ satisfy (8) and $\psi^\alpha(\lambda)$ solve the spectral problem (9). Suppose that* $\det \partial \psi^\alpha(\lambda)/\partial x^\beta \not\equiv 0$*, introduce new independent variables*

$$\tilde{x}^\alpha = \psi^\alpha(\lambda), \tag{11}$$

*and define (locally) new dependent variables $\tilde{K}^\beta$ by the formulae*

$$\frac{\partial \tilde{K}^\beta}{\partial \tilde{x}^\gamma} = \frac{\partial K^\beta}{\partial x^\gamma}. \tag{12}$$

*Then*

$$\tilde{c}^\alpha_{\beta\gamma} = \frac{\partial^2 \tilde{K}^\alpha}{\partial \tilde{x}^\beta \partial \tilde{x}^\gamma}, \tag{13}$$

*where $\tilde{K}^\alpha = \tilde{K}^\alpha(\tilde{x})$ are determined from (12), satisfy*

$$\tilde{c}^\alpha_{\beta\gamma} \tilde{c}^\gamma_{\rho\nu} = \tilde{c}^\alpha_{\nu\gamma} \tilde{c}^\gamma_{\rho\beta}. \tag{14}$$

**Proof.** First, we need to show that (12) is well defined, i.e., there exist, at least locally, the functions $\tilde{K}^\alpha(\tilde{x})$ such that (12) holds. Quite clearly, this amounts to proving that we have

$$\frac{\partial^2 \tilde{K}^\alpha}{\partial \tilde{x}^\beta \partial \tilde{x}^\gamma} = \frac{\partial^2 \tilde{K}^\alpha}{\partial \tilde{x}^\gamma \partial \tilde{x}^\beta}, \tag{15}$$

or equivalently (by virtue of (12)),

$$\frac{\partial^2 K^\alpha}{\partial x^\beta \partial \tilde{x}^\gamma} = \frac{\partial^2 K^\alpha}{\partial x^\gamma \partial \tilde{x}^\beta}. \tag{16}$$

From (12) we have

$$\frac{\partial^2 K^\alpha}{\partial x^\beta \partial \tilde{x}^\gamma} = \frac{\partial^2 K^\alpha}{\partial x^\beta \partial x^\varepsilon} \frac{\partial x^\varepsilon}{\partial \tilde{x}^\gamma} = c^\alpha_{\beta\varepsilon} \frac{\partial x^\varepsilon}{\partial \psi^\gamma}. \tag{17}$$

Hence,

$$\frac{\partial^2 K^\alpha}{\partial x^\beta \partial \tilde{x}^\gamma} - \frac{\partial^2 K^\alpha}{\partial x^\gamma \partial \tilde{x}^\beta} = \left( c^\alpha_{\beta\varepsilon} \frac{\partial x^\varepsilon}{\partial \psi^\gamma} - c^\alpha_{\gamma\varepsilon} \frac{\partial x^\varepsilon}{\partial \psi^\beta} \right). \tag{18}$$

We want to show that the expression on the left-hand side of (18) vanishes. But this is equivalent to vanishing of the following quantity:

$$B^\alpha_{\rho\nu} = \left( c^\alpha_{\beta\varepsilon} \frac{\partial x^\varepsilon}{\partial \psi^\gamma} - c^\alpha_{\gamma\varepsilon} \frac{\partial x^\varepsilon}{\partial \psi^\beta} \right) \frac{\partial \psi^\gamma}{\partial x^\rho} \frac{\partial \psi^\beta}{\partial x^\nu}.$$

Using the obvious identity

$$\frac{\partial \psi^\gamma}{\partial x^\rho} \frac{\partial x^\rho}{\partial \psi^\nu} = \delta^\gamma_\nu,$$

where $\delta_\nu^\gamma$ is the Kronecker delta, we find that

$$B_{\rho\nu}^\alpha = \left( c_{\beta\rho}^\alpha \frac{\partial \psi^\beta}{\partial x^\nu} - c_{\gamma\nu}^\alpha \frac{\partial \psi^\gamma}{\partial x^\rho} \right).$$

Now using (4) we obtain

$$B_{\rho\nu}^\alpha = \lambda \left( c_{\gamma\rho}^\alpha c_{\nu\kappa}^\gamma - c_{\gamma\nu}^\alpha c_{\rho\kappa}^\gamma \right) \psi^\kappa \overset{(1)}{=} 0,$$

and thus

$$\frac{\partial^2 K^\alpha}{\partial x^\beta \partial \tilde{x}^\gamma} - \frac{\partial^2 K^\alpha}{\partial x^\gamma \partial \tilde{x}^\beta} = 0,$$

so (15) indeed holds, i.e.,

$$\tilde{c}_{\beta\gamma}^\alpha = \tilde{c}_{\gamma\beta}^\alpha. \tag{19}$$

Now we only need to prove that (14) holds, or equivalently

$$\tilde{c}_{\beta\gamma}^\alpha \tilde{c}_{\rho\nu}^\gamma - \tilde{c}_{\nu\gamma}^\alpha \tilde{c}_{\rho\beta}^\gamma = 0. \tag{20}$$

Using (19), (17) and (18) we obtain

$$\tilde{c}_{\beta\gamma}^\alpha = \tilde{c}_{\gamma\beta}^\alpha = c_{\gamma\varepsilon}^\alpha \frac{\partial x^\varepsilon}{\partial \psi^\beta} = \frac{\partial x^\varepsilon}{\partial \psi^\beta} c_{\gamma\varepsilon}^\alpha. \tag{21}$$

Using (21) for $\tilde{c}_{\beta\gamma}^\alpha$ in the first term and for $\tilde{c}_{\nu\gamma}^\alpha$ in the second term of (20), and the formula

$$\tilde{c}_{\kappa\rho}^\varepsilon = c_{\kappa\nu}^\varepsilon \frac{\partial x^\nu}{\partial \psi^\rho},$$

for $\tilde{c}_{\rho\nu}^\gamma$ in the first term and for $\tilde{c}_{\rho\beta}^\gamma$ in the second term of (20), we see that the expression on the left-hand side of (20) boils down to

$$\tilde{c}_{\beta\gamma}^\alpha \tilde{c}_{\rho\nu}^\gamma - \tilde{c}_{\nu\gamma}^\alpha \tilde{c}_{\rho\beta}^\gamma = \frac{\partial x^\pi}{\partial \psi^\beta} \frac{\partial x^\kappa}{\partial \psi^\nu} \left( c_{\pi\gamma}^\alpha c_{\rho\kappa}^\gamma - c_{\kappa\gamma}^\alpha c_{\rho\pi}^\gamma \right) \overset{(1)}{=} 0, \tag{22}$$

and thus (14) indeed holds.                                                                                                                                      $\square$

## 4. Nonlocal symmetries for oriented associativity equations

Recall (see e.g. [49, 51–53]) that an (infinitesimal higher) symmetry for the oriented associativity equations (8) is an evolutionary vector field $X = G^\alpha \partial / \partial K^\alpha$ such that $G^\alpha$ satisfy the linearized version of (8), that is,

$$\frac{\partial^2 G^\nu}{\partial x^\alpha \partial x^\rho} \frac{\partial^2 K^\rho}{\partial x^\beta \partial x^\gamma} + \frac{\partial^2 K^\nu}{\partial x^\alpha \partial x^\rho} \frac{\partial^2 G^\rho}{\partial x^\beta \partial x^\gamma} = \frac{\partial^2 G^\rho}{\partial x^\alpha \partial x^\rho} \frac{\partial^2 K^\nu}{\partial x^\beta \partial x^\gamma} + \frac{\partial^2 K^\rho}{\partial x^\alpha \partial x^\beta} \frac{\partial^2 G^\nu}{\partial x^\rho \partial x^\gamma}, \tag{23}$$

modulo (8) and differential consequences thereof (or, informally, on solutions of (8)). This is equivalent to compatibility of (8) with the flow associated with $X$, that is,

$$\partial K^\alpha / \partial \tau = G^\alpha.$$

A straightforward but somewhat tedious computation proves the following generalization of the results of Chen, Kontsevich and Schwarz [40] (see corollary 6 for the latter) to the case of oriented associativity equations.

**Theorem 1.** *The evolutionary vector fields*

$$\psi^\alpha(\lambda) \frac{\partial}{\partial K^\alpha} \qquad and \qquad \psi^\alpha(\lambda) \chi(-\lambda) \frac{\partial}{\partial K^\alpha},$$

*where $\psi^\alpha(\lambda)$ satisfy (9) and $\chi(\lambda)$ satisfies (10), are nonlocal higher symmetries for the oriented associativity equations (8), i.e., the flows*

$$\frac{\partial K^\alpha}{\partial \tau_\lambda} = \psi^\alpha(\lambda), \tag{24}$$

$$\frac{\partial K^\alpha}{\partial \sigma_\lambda} = \psi^\alpha(\lambda)\chi(-\lambda), \tag{25}$$

*are compatible with (8).*

Informally, compatibility here means that the flows (24) and (25) map the set $\mathcal{S}$ of (smooth) solutions of (8) into itself, i.e., $\mathcal{S}$ is invariant under the flows (24) and (25); see e.g. [51–56] and references therein for the general theory of nonlocal symmetries. In a more analytic language, theorem 1 states that $G^\alpha = \psi^\alpha(\lambda)$ and $\tilde{G}^\alpha = \psi^\alpha(\lambda)\chi(-\lambda)$ satisfy (23) provided (8), (9) and (10) hold.

An unusual feature of the symmetries $\psi^\alpha(\lambda)\partial/\partial K^\alpha$ from theorem 1 is that they are linear (rather than quadratic, as it is the case for many other systems, cf [50]) in the solutions of an auxiliary linear problem.

It is natural to ask whether the flows (24) and (25) are integrable systems in any reasonable sense. The following result provides linear spectral problems for these flows and thus suggests their integrability.

**Corollary 1.** *The flows (24) and (25) can be (nonuniquely) extended to the flows for the quantities $\psi^\alpha(\mu)$ and $\chi(\mu)$ as follows:*

$$\frac{\partial \psi^\alpha(\mu)}{\partial \tau_\lambda} = \lambda\mu c^\alpha_{\nu\kappa}\psi^\nu(\lambda)\psi^\kappa(\mu), \tag{26}$$

$$\frac{\partial \chi(\mu)}{\partial \tau_\lambda} = \frac{\lambda\mu}{\lambda+\mu}\psi^\nu(\lambda)\frac{\partial \chi(\mu)}{\partial x^\nu}, \tag{27}$$

$$\frac{\partial \psi^\alpha(\mu)}{\partial \sigma_\lambda} = \lambda\mu c^\alpha_{\nu\kappa}\psi^\nu(\lambda)\psi^\kappa(\mu)\chi(-\lambda) + \frac{\lambda\mu}{\lambda-\mu}\frac{\partial \chi(-\lambda)}{\partial x^\beta}\psi^\beta(\mu)\psi^\alpha(\lambda), \tag{28}$$

$$\frac{\partial \chi(\mu)}{\partial \sigma_\lambda} = \frac{\lambda\mu}{\lambda+\mu}\psi^\nu(\lambda)\frac{\partial \chi(\mu)}{\partial x^\nu}\chi(-\lambda). \tag{29}$$

In particular, by corollary 1 equation (26) together with the system

$$\frac{\partial \psi^\alpha(\mu)}{\partial x^\beta} = \mu\frac{\partial^2 K^\alpha}{\partial x^\beta \partial x^\gamma}\psi^\gamma(\mu), \tag{30}$$

provide (assuming that (9) holds) a zero-curvature representation for the extended system (8), (24), and thus ensure integrability thereof.

Likewise, the flow (25) is integrable because equation (30) along with the system

$$\frac{\partial^2 \chi(\mu)}{\partial x^\alpha \partial x^\beta} = \mu\frac{\partial^2 K^\delta}{\partial x^\alpha \partial x^\beta}\frac{\partial \chi(\mu)}{\partial x^\delta} \tag{31}$$

and (28), (29) provide (assuming that (9) and (10) hold) a linear spectral problem for the extended system (8), (25), i.e., (8) and (25) are precisely the compatibility conditions for (28)–(31), and integrability of the extended system in question follows.

Using the extended flows from corollary 1 we readily obtain the following result:

**Corollary 2.** *All flows (24) and (25) commute for all values of parameters λ and μ:*

$$\frac{\partial^2 K^\alpha}{\partial \tau_\lambda \partial \tau_\mu} = \frac{\partial^2 K^\alpha}{\partial \tau_\mu \partial \tau_\lambda}, \qquad \frac{\partial^2 K^\alpha}{\partial \tau_\lambda \partial \sigma_\mu} = \frac{\partial^2 K^\alpha}{\partial \sigma_\mu \partial \tau_\lambda}, \qquad \frac{\partial^2 K^\alpha}{\partial \sigma_\lambda \partial \sigma_\mu} = \frac{\partial^2 K^\alpha}{\partial \sigma_\mu \partial \sigma_\lambda}.$$

It is important to stress that this result *per se* does *not* imply commutativity of the *extended* flows from corollary 1. However, a straightforward computation proves the following assertion.

**Corollary 3.** *The extended flows (24), (26), (27) commute, i.e., for all values of λ, μ and ζ we have*

$$\frac{\partial^2 K^\alpha}{\partial \tau_\lambda \partial \tau_\mu} = \frac{\partial^2 K^\alpha}{\partial \tau_\mu \partial \tau_\lambda}, \qquad \frac{\partial^2 \psi^\alpha(\zeta)}{\partial \tau_\lambda \partial \tau_\mu} = \frac{\partial^2 \psi^\alpha(\zeta)}{\partial \tau_\mu \partial \tau_\lambda}, \qquad \frac{\partial^2 \chi(\zeta)}{\partial \tau_\lambda \partial \tau_\mu} = \frac{\partial^2 \chi(\zeta)}{\partial \tau_\mu \partial \tau_\lambda}. \tag{32}$$

We intend to study the remaining commutation relations for the extended flows in more detail elsewhere.

## 5. Expansion in the spectral parameter and nonlocal potentials

Now consider a formal Taylor expansion for $\psi^\alpha$ in λ,

$$\psi^\alpha(\lambda) = \sum_{k=0}^\infty \psi_k^\alpha \lambda^k. \tag{33}$$

It is immediate from theorem 1 that $\psi_k^\alpha \partial/\partial K^\alpha$ are symmetries for (8), i.e., the flows

$$\frac{\partial K^\alpha}{\partial \tau_k} = \psi_k^\alpha, \qquad k = 0, 1, 2, \ldots, \tag{34}$$

are compatible with (8).

We readily find from (9) the following recursion relation:

$$\frac{\partial \psi_k^\alpha}{\partial x^\beta} = \frac{\partial^2 K^\alpha}{\partial x^\beta \partial x^\gamma} \psi_{k-1}^\gamma, \qquad k = 1, 2, \ldots. \tag{35}$$

For $k = 0$ we have $\partial \psi_0^\alpha / \partial x^\beta = 0$ for all $\beta = 1, \ldots, n$.

Now let $(w_0)_\beta^\alpha = \delta_\beta^\alpha$, where $\delta_\beta^\alpha$ is the Kronecker delta, and $(w_1)_\beta^\alpha = \partial K^\alpha / \partial x^\beta$. Define recursively the following sequence of nonlocal quantities:

$$\frac{\partial (w_k)_\gamma^\beta}{\partial x^\alpha} = \frac{\partial^2 K^\beta}{\partial x^\alpha \partial x^\rho} (w_{k-1})_\gamma^\rho, \qquad k = 2, 3, \ldots. \tag{36}$$

We have

$$\psi_k^\alpha = \sum_{j=0}^k h_j^\gamma (w_{k-j})_\gamma^\alpha, \qquad k = 0, 1, 2, \ldots, \tag{37}$$

where $h_j^\gamma$ are arbitrary constants.

In analogy with (33), consider a formal Taylor expansion for $\chi$ in λ,

$$\chi(\lambda) = \sum_{k=0}^\infty \chi_k \lambda^k. \tag{38}$$

We obtain from (10) the following recursion relation:

$$\frac{\partial^2 \chi_k}{\partial x^\alpha \partial x^\gamma} = \frac{\partial^2 K^\nu}{\partial x^\alpha \partial x^\gamma} \frac{\partial \chi_{k-1}}{\partial x^\nu}, \qquad k = 1, 2, \ldots. \tag{39}$$

Set $v_0^\alpha = x^\alpha$, $v_1^\alpha = K^\alpha$, and, in analogy with (6) and (36), define the following sequence of nonlocal quantities $v_k^\beta$, $\beta = 1, \ldots, n$:

$$\frac{\partial^2 v_k^\beta}{\partial x^\alpha \partial x^\gamma} = \frac{\partial^2 K^\nu}{\partial x^\alpha \partial x^\gamma} \frac{\partial v_{k-1}^\beta}{\partial x^\nu}, \qquad k = 2, 3, \ldots. \tag{40}$$

In terms of geometric theory of PDEs, see e.g. [51, 52, 55, 56], the quantities $(w_k)_\beta^\alpha$ and $v_k^\gamma$, $\alpha, \beta, \gamma = 1, \ldots, n, k = 2, 3, \ldots$, define an infinite-dimensional Abelian covering over (8).

We have the following counterpart of (37):

$$\chi_k = b_k + \sum_{j=0}^{k} d_{k-j,\gamma} v_j^\gamma, \qquad k = 0, 1, 2, \ldots, \tag{41}$$

where $b_k$ and $d_{j,\gamma}$ are arbitrary constants.

Using (33) and (38) we readily find that

$$\psi^\alpha(\lambda)\chi(-\lambda) = \sum_{k=0}^{\infty} \rho_k^\alpha \lambda^k, \qquad \rho_k^\alpha \overset{\text{def}}{=} \sum_{j=0}^{k} (-1)^j \chi_j \psi_{k-j}^\alpha.$$

It is now immediate from theorem 1 that $(w_k)_\beta^\alpha \partial/\partial K^\alpha$ and $\rho_k^\alpha \partial/\partial K^\alpha$ are symmetries for (8), and for $k \geq 2$ these symmetries are nonlocal.

What is more, using corollary 1 we readily obtain the following result.

**Corollary 4.** *The oriented associativity equations (8) have infinitely many symmetries of the form*

$$X_{k,\beta} = (w_k)_\beta^\alpha \frac{\partial}{\partial K^\alpha} \qquad and \qquad Y_{k,\gamma}^\beta = \sum_{j=0}^{k} (-1)^j v_j^\beta \cdot (w_{k-j})_\gamma^\alpha \frac{\partial}{\partial K^\alpha},$$

*and all associated flows, i.e.,*

$$\frac{\partial K^\alpha}{\partial \tau_k^\beta} = (w_k)_\beta^\alpha, \qquad \frac{\partial K^\alpha}{\partial \sigma_{k,\beta}^\gamma} = \sum_{j=0}^{k} (-1)^j v_j^\beta \cdot (w_{k-j})_\gamma^\alpha,$$

*commute*

$$\frac{\partial^2 K^\alpha}{\partial \tau_k^\beta \partial \tau_l^\nu} = \frac{\partial^2 K^\alpha}{\partial \tau_l^\nu \partial \tau_k^\beta}, \qquad \frac{\partial^2 K^\alpha}{\partial \sigma_{k,\gamma}^\beta \partial \tau_{l,\nu}^\rho} = \frac{\partial^2 K^\alpha}{\partial \tau_{l,\nu}^\rho \partial \sigma_{k,\gamma}^\beta}, \qquad \frac{\partial^2 K^\alpha}{\partial \sigma_{k,\gamma}^\beta \partial \sigma_{l,\nu}^\rho} = \frac{\partial^2 K^\alpha}{\partial \sigma_{l,\nu}^\rho \partial \sigma_{k,\gamma}^\beta},$$

$$k, l = 0, 1, 2, \ldots, \qquad \alpha, \beta, \gamma, \delta, \nu, \rho = 1, \ldots, n.$$

It is readily seen that the symmetries $X_{k,\beta}$ and $Y_{k,\gamma}^\beta$ and the associated flows are nonlocal for $k \geq 2$. We stress once more that the part of corollary 4 about commutativity of the flows associated with $X_{k,\beta}$ and $Y_{k,\gamma}^\beta$ makes substantial use of the extended flows from corollary 1 which are not uniquely defined. Moreover, corollary 4 does *not* imply commutativity of the *extended* flows associated with the symmetries $X_{k,\beta}$ and $Y_{k,\gamma}^\beta$. However, the flows associated with the symmetries $X_{k,\beta}$ do commute after a (suitable) extension to the variables $(w_k)_\beta^\alpha$. Namely, using corollaries 1 and 3 we readily arrive at the following assertion.

**Corollary 5.** *The flows*

$$\frac{\partial K^\alpha}{\partial \tau_k^\beta} = (w_k)_\beta^\alpha,$$

$$\frac{\partial (w_l)_\pi^\alpha}{\partial \tau_k^\beta} = c_{\nu\rho}^\alpha (w_{k-1})_\beta^\nu (w_{l-1})_\pi^\rho, \qquad l = 0, 1, 2, \ldots, \tag{42}$$

*where* $(w_0)^\alpha_\beta = \delta^\alpha_\beta$, $(w_1)^\alpha_\beta = \partial K^\alpha / \partial x^\beta$, *and we set* $(w_{-1})^\nu_\beta \equiv 0$ *for convenience, commute for all* $\beta, \gamma = 1, \ldots, n$ *and all* $k, l = 0, 1, 2, \ldots$.

*Equivalently, the evolutionary vector fields*

$$\bar{X}_{s,\beta} \equiv X_{s,\beta} = (w_s)^\alpha_\beta \frac{\partial}{\partial K^\alpha}, \qquad s = 0,$$

$$\bar{X}_{k,\beta} = (w_k)^\alpha_\beta \frac{\partial}{\partial K^\alpha} + \sum_{l=2}^{\infty} c^\alpha_{\kappa\rho} (w_{k-1})^\kappa_\beta (w_{l-1})^\rho_\pi \frac{\partial}{\partial (w_l)^\alpha_\pi}, \qquad k = 1, 2, 3, \ldots,$$

*commute, i.e.,* $[\bar{X}_{k,\beta}, \bar{X}_{l,\gamma}] = 0$ *(see e.g. [51] for the definition of the bracket* [, ]*), for all* $\beta, \gamma = 1, \ldots, n$ *and all* $k, l = 0, 1, 2, \ldots$.

Thus, the oriented associativity equations (8) possess an infinite hierarchy of commuting flows whose existence reconfirms integrability of (8).

## 6. Nonlocal symmetries for the gradient reduction of oriented associativity equations

Following [14], consider the so-called gradient reduction of (8). Namely, assume that there exist a nondegenerate symmetric constant matrix $\eta^{\alpha\beta}$ and a function $F = F(x^1, \ldots, x^n)$, known as a prepotential in 2D topological field theories [1–3], such that

$$K^\alpha = \eta^{\alpha\beta} \partial F / \partial x^\beta. \tag{43}$$

Then (8) boils down to the famous associativity equations for $F$ [1–4]

$$\frac{\partial^3 F}{\partial x^\alpha \partial x^\beta \partial x^\delta} \eta^{\delta\gamma} \frac{\partial^3 F}{\partial x^\gamma \partial x^\nu \partial x^\rho} = \frac{\partial^3 F}{\partial x^\alpha \partial x^\nu \partial x^\delta} \eta^{\delta\gamma} \frac{\partial^3 F}{\partial x^\gamma \partial x^\beta \partial x^\rho}. \tag{44}$$

Note that in the standard theory of the WDVV equations (see e.g. [1–4]) it is further required that

$$\frac{\partial^3 F}{\partial x^\alpha \partial x^\beta \partial x^1} = \eta_{\alpha\beta}, \tag{45}$$

where $\eta_{\alpha\beta}$ is a nondegenerate constant matrix such that $\eta_{\alpha\beta}\eta^{\beta\gamma} = \delta^\gamma_\alpha$.

However, in what follows we shall not impose (45) and the so-called quasihomogeneity condition (see e.g. [1–4, 7] for the discussion of these conditions).

Upon assuming (43) we find that the auxiliary linear problem (9) also admits a reduction

$$\psi^\alpha = \eta^{\alpha\beta} \partial \chi / \partial x^\beta. \tag{46}$$

This, along with (43), turns (9) into the following overdetermined system of the Gauss–Manin equations for $\chi$:

$$\frac{\partial^2 \chi(\lambda)}{\partial x^\alpha \partial x^\gamma} = \lambda \eta^{\rho\nu} \frac{\partial^3 F}{\partial x^\alpha \partial x^\gamma \partial x^\rho} \frac{\partial \chi(\lambda)}{\partial x^\nu}. \tag{47}$$

This is nothing but (10) after the substitution (43), and again the associativity equations (44) are nothing but the compatibility conditions for (47); see e.g. [3, 4, 30] for the discussion of geometric aspects of (47).

Using theorem 1 in conjunction with (43) and (46) we recover the following result from [40].

**Corollary 6.** *For any solution* $\chi(\lambda)$ *of* (47) *the quantities*

$$\chi(\lambda) \frac{\partial}{\partial F} \qquad and \qquad \chi(\lambda) \chi(-\lambda) \frac{\partial}{\partial F}$$

*are nonlocal higher symmetries for the associativity equations (44), i.e., the equations*

$$\frac{\partial F}{\partial \tau_\lambda} = \chi(\lambda), \tag{48}$$

$$\frac{\partial F}{\partial \zeta_\lambda} = \chi(\lambda)\chi(-\lambda) \tag{49}$$

*are compatible with (44).*

The above flows can be (nonuniquely) extended as follows

$$\frac{\partial \chi(\mu)}{\partial \tau_\lambda} = \frac{\lambda\mu}{\lambda+\mu}\eta^{\nu\beta}\frac{\partial \chi(\lambda)}{\partial x^\beta}\frac{\partial \chi(\mu)}{\partial x^\nu}, \tag{50}$$

$$\frac{\partial \chi(\mu)}{\partial \zeta_\lambda} = \frac{\lambda\mu}{\lambda+\mu}\eta^{\nu\beta}\frac{\partial \chi(\lambda)}{\partial x^\beta}\frac{\partial \chi(\mu)}{\partial x^\nu}\chi(-\lambda) + \frac{\lambda\mu}{\lambda-\mu}\eta^{\nu\beta}\frac{\partial \chi(-\lambda)}{\partial x^\beta}\frac{\partial \chi(\mu)}{\partial x^\nu}\chi(\lambda). \tag{51}$$

In particular, this result means that $\chi(\lambda)$ and $\chi(\lambda)\chi(-\lambda)$ satisfy the linearized version of (44) provided (44) and (47) hold. Using the extended flows from corollary 6 we readily obtain

**Corollary 7.** *All flows (48) and (49) commute: for all values of parameters $\lambda$ and $\mu$ we have*

$$\frac{\partial^2 F}{\partial \tau_\lambda \partial \tau_\mu} = \frac{\partial^2 F}{\partial \tau_\mu \partial \tau_\lambda}, \qquad \frac{\partial^2 F}{\partial \tau_\lambda \partial \zeta_\mu} = \frac{\partial^2 F}{\partial \zeta_\mu \partial \tau_\lambda}, \qquad \frac{\partial^2 F}{\partial \zeta_\lambda \partial \zeta_\mu} = \frac{\partial^2 F}{\partial \zeta_\mu \partial \zeta_\lambda}.$$

Perhaps a bit surprisingly, the proper counterpart of the flow (49) for the oriented associativity equations (8) is not (25) itself but a linear combination of the flows (25) with the opposite values of $\lambda$,

$$\frac{\partial K^\alpha}{\partial \zeta_\lambda} = \psi^\alpha(\lambda)\chi(-\lambda) + \psi^\alpha(-\lambda)\chi(\lambda).$$

Consider now a formal Taylor expansion in $\lambda$ for a solution $\chi(\lambda)$ of (47),

$$\chi(\lambda) = \sum_{k=0}^{\infty} \chi_k \lambda^k.$$

Note that using a slightly different expansion of $\chi(\lambda)$, involving also $\lambda^{-1}$, enables one to construct solutions of the WDVV equations directly from $\chi(\lambda)$, see [58] and references therein.

Equations (41) remain valid when $\chi(\lambda)$ satisfies (47) instead of (10) if we substitute $\eta^{\alpha\beta}\partial F/\partial x^\beta$ for $K^\alpha$ into the definitions of $v_k^\alpha$ and (41). Then expanding the symmetries from corollary 6 in powers of $\lambda$ yields

**Corollary 8.** *The associativity equations (44) have infinitely many symmetries of the form*

$$\tilde{X}_k^\beta = v_k^\beta \frac{\partial}{\partial F}, \quad and \quad \tilde{Z}_k^{\alpha\beta} = \sum_{j=0}^{k}(-1)^j v_j^\alpha v_{k-j}^\beta \frac{\partial}{\partial F}, \quad k = 0,1,2,\ldots, \quad \alpha,\beta = 1,\ldots,n,$$

*and all associated flows, i.e.,*

$$\frac{\partial F}{\partial \tau_{\beta,k}} = v_k^\beta, \qquad \frac{\partial F}{\partial \zeta_{\alpha\beta,k}} = \sum_{j=0}^{k}(-1)^j v_j^\alpha v_{k-j}^\beta, \tag{52}$$

*commute:*

$$\frac{\partial^2 F}{\partial \tau_{\beta,k} \partial \tau_{\nu,l}} = \frac{\partial^2 F}{\partial \tau_{\nu,l} \partial \tau_{\beta,k}}, \qquad \frac{\partial^2 F}{\partial \tau_{\gamma,k} \partial \zeta_{\alpha\beta,l}} = \frac{\partial^2 F}{\partial \zeta_{\alpha\beta,l} \partial \tau_{\gamma,k}}, \qquad \frac{\partial^2 F}{\partial \zeta_{\rho\nu,k} \partial \zeta_{\alpha\beta,l}} = \frac{\partial^2 F}{\partial \zeta_{\alpha\beta,l} \partial \zeta_{\rho\nu,k}},$$

$$k, l = 0, 1, 2, \ldots, \qquad \alpha, \beta, \gamma, \delta, \nu, \rho = 1, \ldots, n.$$

Again, it is clear that the symmetries $\tilde{X}_k^\beta$ and $\tilde{Z}_k^{\alpha\beta}$ and the associated flows are nonlocal for $k \geqslant 2$.

Note that the above results do *not* imply commutativity for *all* of the *extended* flows from corollary 6 and therefore do not contradict the results from [39, 40]. On the other hand, using corollary 3 we readily find that, in perfect agreement with [40], the extended flows (48), (50) do commute, i.e.,

$$\frac{\partial^2 F}{\partial \tau_\lambda \partial \tau_{\lambda'}} = \frac{\partial^2 F}{\partial \tau_{\lambda'} \partial \tau_\lambda}, \qquad \frac{\partial^2 \chi(\mu)}{\partial \tau_\lambda \partial \tau_{\lambda'}} = \frac{\partial^2 \chi(\mu)}{\partial \tau_{\lambda'} \partial \tau_\lambda}, \tag{53}$$

for all values of $\lambda$, $\lambda'$ and $\mu$.

The quantities $\chi_k$ coincide, up to a choice of normalization, with the densities of Hamiltonians of integrable bi-Hamiltonian hydrodynamic-type systems associated with any solution of the WDVV equations, see lecture 6 of [4] and, e.g., [57], and references therein. It was mentioned in [4] that it is natural to consider these hydrodynamic-type systems as higher (Lie–Bäcklund) symmetries for the WDVV equations because using these systems one can construct [4] the Bäcklund transformation for the WDVV equations. We have now seen that $\chi_k$ and $v_k^\alpha$ can also be interpreted as symmetries of the associativity equations (44) (and, upon imposing necessary restrictions on $F$ and $\chi$, of the WDVV equations) in a far more straightforward manner.

## 7. Intermediate integrals and Bäcklund-type transformations

The compatibility conditions

$$\frac{\partial^2 (w_2)_\gamma^\beta}{\partial x^\alpha \partial x^\nu} = \frac{\partial^2 (w_2)_\gamma^\beta}{\partial x^\nu \partial x^\alpha}$$

for (36) with $k = 2$ yield precisely equations (8), and we have

**Corollary 9.** *If the functions $K^\alpha$ and $G_\gamma^\beta$ satisfy the system*

$$\frac{\partial^2 K^\beta}{\partial x^\alpha \partial x^\rho} \frac{\partial K^\rho}{\partial x^\gamma} = \frac{\partial G_\gamma^\beta}{\partial x^\alpha}, \tag{54}$$

*then $K^\alpha$ automatically satisfy the oriented associativity equations (8).*

*In particular, if under the above assumptions $K^\alpha = \eta^{\alpha\beta} \partial F / \partial x^\beta$, where $\eta^{\alpha\beta}$ is a symmetric nondegenerate constant matrix, then F automatically satisfies the associativity equations (44).*

Just like (54), the compatibility conditions

$$\frac{\partial^3 v_2^\nu}{\partial x^\alpha \partial x^\beta \partial x^\gamma} = \frac{\partial^3 v_2^\nu}{\partial x^\gamma \partial x^\beta \partial x^\alpha}$$

for (40) with $k = 2$ also yield nothing but equations (8), and we obtain

**Corollary 10.** *If the functions $K^\alpha$ and $G^\beta$ satisfy*

$$\frac{\partial^2 K^\nu}{\partial x^\alpha \partial x^\gamma} \frac{\partial K^\beta}{\partial x^\nu} = \frac{\partial^2 G^\beta}{\partial x^\alpha \partial x^\gamma}, \tag{55}$$

*then $K^\alpha$ automatically satisfy the oriented associativity equations (8).*

*In particular, if under the above assumptions $K^\alpha = \eta^{\alpha\beta} \partial F / \partial x^\beta$, where $\eta^{\alpha\beta}$ is a symmetric nondegenerate constant matrix, then F automatically satisfies the associativity equations (44).*

Thus, (54) and (55) yield a kind of intermediate integrals for (8) (or, if $K^\alpha = \eta^{\alpha\beta} \partial F / \partial x^\beta$, for (44)). In the terminology of [51, 52, 55, 56], (54) and (55) also define coverings over (8) (respectively, if $K^\alpha = \eta^{\alpha\beta} \partial F / \partial x^\beta$, over (44)): for any solution $K^\alpha$ of (8) (respectively for any solution F of (44)) there exist, at least locally, the functions $G^\alpha_\beta$ and $G^\alpha$ such that (54) and (55) hold.

Moreover, we have the following observation.

**Corollary 11.** *If the functions $K^\alpha$ and $H^\alpha$ satisfy the system*

$$\frac{\partial^2 K^\beta}{\partial x^\alpha \partial x^\rho} \frac{\partial K^\rho}{\partial x^\gamma} = \frac{\partial^2 H^\beta}{\partial x^\alpha \partial x^\gamma}, \tag{56}$$

*then $K^\alpha$ and $\tilde{K}^\alpha = H^\alpha$ automatically satisfy the oriented associativity equations (8).*

Hence, (56) provides a conditional Bäcklund transformation for (8): if $K^\alpha$ satisfy (8) *and* the conditions

$$\frac{\partial^2 K^\beta}{\partial x^\alpha \partial x^\rho} \frac{\partial K^\rho}{\partial x^\gamma} = \frac{\partial^2 K^\beta}{\partial x^\gamma \partial x^\rho} \frac{\partial K^\rho}{\partial x^\alpha} \tag{57}$$

which are necessary for (56) to hold, then there exist, at least locally, $\tilde{K}^\alpha = H^\alpha$ such that (56) holds and, moreover, these $\tilde{K}^\alpha$ also satisfy (8).

Setting $K^\alpha = \eta^{\alpha\beta} \partial F / \partial x^\beta$, where $\eta^{\alpha\beta}$ is a symmetric nondegenerate constant matrix, in corollary 11 yields

**Corollary 12.** *Let the functions F and $H^\alpha$ satisfy the system*

$$\eta^{\beta\nu} \eta^{\rho\kappa} \frac{\partial^3 F}{\partial x^\alpha \partial x^\rho \partial x^\nu} \frac{\partial^2 F}{\partial x^\gamma \partial x^\kappa} = \frac{\partial^2 H^\beta}{\partial x^\alpha \partial x^\gamma}. \tag{58}$$

*Then F automatically satisfies the associativity equations (44) and $\tilde{K}^\alpha = H^\alpha$ automatically satisfy the oriented associativity equations (8).*

In complete analogy with the above, (58) provides a Bäcklund transformation relating the associativity equations (44) supplemented with the conditions

$$\eta^{\rho\kappa} \frac{\partial^3 F}{\partial x^\alpha \partial x^\rho \partial x^\nu} \frac{\partial^2 F}{\partial x^\gamma \partial x^\kappa} = \eta^{\rho\kappa} \frac{\partial^3 F}{\partial x^\gamma \partial x^\rho \partial x^\nu} \frac{\partial^2 F}{\partial x^\alpha \partial x^\kappa} \tag{59}$$

which are necessary for (58) to hold, and the oriented associativity equations (8) for $\tilde{K}^\alpha = H^\alpha$.

Note that the system (59) was originally found and studied in [20, 22, 23] (in [20] it was referred to as a condition for compatible potential deformation of a pair of the Frobenius algebras) because this system plays an important role in the classification of compatible Hamiltonian structures of hydrodynamic type. Moreover, in [59] (cf also [29]) it was proved that (59) is also equivalent to the condition of involutivity for a certain set of functionals constructed from the function F with respect to the constant homogeneous first-order Poisson bracket of hydrodynamic type associated with the flat contravariant metric $\eta^{\alpha\beta}$.

## 8. Conclusions and open problems

In the present paper we have found infinite hierarchies of nonlocal higher symmetries for the oriented and 'usual' associativity equations (8) and (44). These symmetries can be employed for producing new solutions from the known ones and for constructing invariant solutions using the standard techniques as presented, e.g., in [49, 51, 52].

Moreover, it is natural to ask whether there exist nonlocal symmetries and conservation laws for (8) (respectively for (44)) that depend on the nonlocal variables (36), (40) (respectively (40) with $K^\alpha = \eta^{\alpha\beta}\partial F/\partial x^\beta$) in a more complicated fashion than the symmetries found in corollary 4 (respectively corollary 8). For instance, one could look for potential (in the sense of [53, 60] and references therein) symmetries and conservation laws of (8) involving the nonlocal variables (36) and (40).

The next steps to take include elucidating the relationship among the nonlocal symmetries of (44) from corollary 6 and the symmetries found in [37] for the generalized (in the sense of [42]) WDVV equations. The relationship (if any exists) of the flows (52) to the flows (5.15) from [61] could be of interest too. Understanding the precise relationship of the symmetries from corollary 8 to the tau-function and the Bäcklund transformations for the WDVV equations from [4] is yet another challenge. It would be also interesting to find recursion operators or master symmetries for (8) and (44) that generate the hierarchies from corollaries 4 and 8.

Finally, it remains to be seen whether the Darboux-type transformation from section 3 and the results from section 7 could indeed yield new solutions for the oriented and 'usual' associativity equations.

We intend to address some of the above issues in our future work.

## Acknowledgments

## References

[1] Witten E 1990 On the structure of topological phase of two-dimensional gravity *Nucl. Phys.* B **340** 281–332
[2] Dijkgraaf R, Verlinde H and Verlinde E 1991 Topological strings in $d < 1$ *Nucl. Phys.* B **352** 59–86
[3] Dubrovin B 1992 Integrable systems in topological field theory *Nucl. Phys.* B **379** 627–89
[4] Dubrovin B 1996 Geometry on 2D topological field theories *Integrable Systems and Quantum Groups (Montecatini Terme, Italy, 1993)* (*Lect. Notes Math.* vol 1620) (Berlin: Springer) pp 120–348 arXiv:hep-th/9407018
[5] Kontsevich M and Manin Yu 1994 Gromov–Witten classes, quantum cohomology, and enumerative geometry *Commun. Math. Phys.* **164** 525–62 (arXiv:hep-th/9402147)
[6] Dubrovin B 1993 Geometry and integrability of topological–antitopological fusion *Commun. Math. Phys.* **152** 539–64 (arXiv:hep-th/9206037)
[7] Manin Y I 1999 *Frobenius Manifolds, Quantum Cohomology and Moduli Spaces* (Providence, RI: AMS)
[8] Hertling C 2002 *Frobenius Manifolds and Moduli Spaces for Singularities* (Cambridge: Cambridge University Press)
[9] Manin Y I 2006 Manifolds with multiplication on the tangent sheaf *Rend. Mat. Appl.* **26** 69–85 (arXiv:math/0502578v1[math.AG])

[10] Hertling C and Manin Y I 1999 Weak Frobenius manifolds *Int. Math. Res. Notices* **6** 277–86 (arXiv:math.QA/9810132)

[11] Losev A and Manin Y I 2004 Extended modular operads *Frobenius Manifolds. Quantum Cohomology and Singularities Proc. Workshop (Bonn, July 8–19, 2002) (Aspects of Mathematics, E36. Friedr)* ed C Hertling and M Marcolli (Wiesbaden: Vieweg & Sohn) pp 181–211 (arXiv:math/0301003000 [math.AG])

[12] Losev A and Polyubin I 2003 Commutativity equations and dressing transformations *JETP Lett.* **77** 53–7

[13] Manin Y I 2005 F-manifolds with flat structure and Dubrovin's duality *Adv. Math.* **198** 5–26 (arXiv:math/0402451v1 [math.DG])

[14] Konopelchenko B G 2009 Quantum deformations of associative algebras and integrable systems *J. Phys. A: Math. Theor.* **42** 095201 (arXiv:0802.3022)

[15] Konopelchenko B G 2008 On the deformation theory of structure constants for associative algebras arXiv:0811.4725

[16] Konopelchenko B G and Magri F 2007 Coisotropic deformations of associative algebras and integrable systems *Commun. Math. Phys.* **274** 627–58 (arXiv:nlin/0606069)

[17] Konopelchenko B G and Magri F 2007 Dispersionless integrable equations as coisotropic deformations: extensions and reductions *Theor. Math. Phys.* **151** 803–19 (arXiv:nlin/0608010)

[18] Konopelchenko B G 2008 Discrete, q-difference deformations of associative algebras and integrable systems arXiv:0809.1938

[19] Konopelchenko B G 2008 Continuous–discrete integrable equations and Darboux transformations as deformations of associative algebras arXiv:0809.3852

[20] Mokhov O I 1998 On compatible potential deformations of Frobenius algebras and associativity equations *Russ. Math. Surv.* **53** 396–7

[21] Kito H 1999 On Hessian structures of the Euclidean space and hyperbolic space *Osaka J. Math.* **36** 51–62 available at http://projecteuclid.org/euclid.ojm/1200788447

[22] Mokhov O I 1999 Compatible Poisson structures of hydrodynamic type and the associativity equations in two-dimensional topological field theory *Rep. Math. Phys.* **43** 247–56

[23] Mokhov O I 1999 Compatible Poisson structures of hydrodynamic type and associativity equations *Proc. Steklov Inst. Math.* **225** 269–84

[24] Dubrovin B 2004 On almost duality for Frobenius manifolds *Geometry, Topology and Mathematical Physics* (*Am. Math. Soc. Trans.* vol 212) (Providence: AMS) pp 75–132 (arXiv:math/0307374v2 [math.DG])

[25] Witten E 1991 Two-dimensional gravity and intersection theory on moduli space *Surveys in Differential Geometry Cambridge, MA, 1990)* (Bethlehem, PA: Lehigh Univ.) pp 243–310

[26] Mokhov O I 2004 *Symplectic and Poisson Geometry on Loop Spaces of Smooth Manifolds and Integrable Equations* (Moscow: Inst. of Computer Studies) (in Russian)

Mokhov O I 2001 *Symplectic and Poisson Geometry on Loop Spaces of Smooth Manifolds and Integrable Equations (Reviews in Mathematics and Mathematical Physics* vol 11, part 2*)* (Amsterdam: Harwood Academic) (earlier Engl. edn)

[27] Mokhov O I 2006 Nonlocal Hamiltonian operators of hydrodynamic type with flat metrics, integrable hierarchies, and associativity equations *Funct. Anal. Appl.* **40** 11–23 (arXiv:math/0406292)

[28] Mokhov O I 2007 Theory of submanifolds, associativity equations in 2D topological quantum field theories, and Frobenius manifolds *Theor. Math. Phys.* **152** 1183–90 (arXiv:math/0610933)

[29] Mokhov O I 2008 Frobenius manifolds as a special class of submanifolds in pseudo-Euclidean spaces *Geometry, Topology, and Mathematical Physics. S P Novikov's Seminar: 2006-2007* (Providence, RI: American Mathematical Society) pp 213–46 (arXiv:0710.5860)

[30] Ferapontov E V 2004 Hypersurfaces with flat centroaffine metric and equations of associativity *Geometriae Dedicata* **103** 33–49 (arXiv:math.DG/0205248)

[31] Strachan I A B 2004 Frobenius manifolds: natural submanifolds and induced bi-Hamiltonian structures *Differ. Geom. Appl.* **20** 67–99 (arXiv:math/0201039)

[32] David L and Strachan I A B 2004 Compatible metrics on a manifold and nonlocal bi-Hamiltonian structures *Int. Math. Res. Notices* **2004** 3533–57 (arXiv:math/0404410)

[33] David L and Strachan I A B 2006 Conformally flat pencils of metrics, Frobenius structures and a modified Saito construction *J. Geom. Phys.* **56** 1561–75 (arXiv:math/0502128)

[34] Pavlov M V 2007 Integrability of Egorov systems of hydrodynamic type *Theor. Math. Phys.* **150** 225–43 (arXiv:nlin/0606017)

[35] Pavlov M V 2006 New Hamiltonian formalism and Lagrangian representations for integrable hydrodynamic type systems arXiv:nlin.SI/0608029

[36] Boyarsky A, Marshakov A, Ruchayskiy O, Wiegmann P and Zabrodin A 2001 On associativity equations in dispersionless integrable hierarchies *Phys. Lett.* B **515** 483–92 (arXiv:hep-th/0105260)

[37] Mironov A and Morozov A 1998 Covariance of WDVV equations *Phys. Lett.* B **424** 48–52 (arXiv:hep-th/9712177)

[38] Geurts M L, Martini R and Post G F 2002 Symmetries of the WDVV equations *Acta Appl. Math.* **72** 67–75

[39] Givental A 2004 Symplectic geometry of Frobenius structures *Frobenius Manifolds. Quantum Cohomology and Singularities, Proc. Workshop (Bonn, July 8–19 2002) (Aspects of Mathematics, E36. Friedr.)* ed C Hertling and M Marcolli (Wiesbaden: Vieweg & Sohn) pp 91–112 (arXiv:math.AG/0305409)

[40] Chen Y, Kontsevich M and Schwarz A 2005 Symmetries of WDVV equations *Nucl. Phys.* B **730** 352–63 (arXiv:hep-th/0508221)

[41] Conte R and Gandarias M L 2005 Symmetry reductions of a particular set of equations of associativity in two-dimensional topological field theory *J. Phys. A: Math. Gen.* **38** 1187–96 (arXiv:math/0412439)

[42] Morozov A 1998 On integrable structure behind the generalized WDVV equations *Phys. Lett.* B **427** 93–6 (arXiv:hep-th/9711194)

[43] de Wit B and Marshakov A 2001 Electric–magnetic duality and WDVV equations *Theor. Math. Phys.* **129** 1504–10 (arXiv:hep-th/0105289)

[44] Hoevenaars L K 2004 Duality transformations for generalized WDVV equations in Seiberg–Witten theory *Phys. Lett.* B **601** 214–21 (arXiv:hep-th/0202007)

[45] Riley A and Strachan I A B 2007 A note on the relationship between rational and trigonometric solutions of the WDVV equations *J. Nonlinear Math. Phys.* **14** 82–94 (arXiv:nlin/0605005 [nlin.SI])

[46] Ferapontov E V, Galvao C A P, Mokhov O I and Nutku Y 1997 Bi-Hamiltonian structure of equations of associativity in 2-d topological field theory *Commun. Math. Phys.* **186** 649–69

[47] Ferapontov E V and Mokhov O I 1997 On the Hamiltonian representation of the associativity equations *Algebraic Aspects of Integrable Systems (Progr. Nonlinear Differential Equations Appl.* vol 26) (Boston, MA: Birkhäuser) 75–91

[48] Ferapontov E V and Mokhov O I 1997 The associativity equations in the two-dimensional topological field theory as integrable Hamiltonian nondiagonalizable systems of hydrodynamic type *Funct. Anal. Appl.* **30** 195–203 (arXiv:hep-th/9505180)

[49] Olver P J 1993 *Applications of Lie Groups to Differential Equations* 2nd edn (*Graduate Texts in Mathematics* vol 107) (New York: Springer)

[50] Konopelchenko B G 1987 *Nonlinear Integrable Equations. Recursion Operators, Group-Theoretical and Hamiltonian Structures of Soliton Equations* (*Lect. Notes Phys.* vol 270) (Berlin: Springer)

[51] Bocharov A V *et al* 1999 *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics* (Providence, RI: AMS)

[52] Krasil'shchik I S and Kersten P H M 2000 *Symmetries and Recursion Operators for Classical and Supersymmetric Differential Equations* (Dordrecht: Kluwer)

[53] Bluman G W and Kumei S 1989 *Symmetries and Differential Equations* (New York: Springer)

[54] Konopelchenko B G and Mokhnachev V G 1979 On the group-theoretical analysis of differential equations *Sov. J. Nucl. Phys.* **30** 559–67

[55] Krasil'shchik I S and Vinogradov A M 1984 Nonlocal symmetries and the theory of coverings: an addendum to A M Vinogradov's 'Local symmetries and conservation laws' *Acta Appl. Math.* **2** 79–96

[56] Krasil'shchik I S and Vinogradov A M 1989 Nonlocal trends in the geometry of differential equations: symmetries, conservation laws, and Bäcklund transformations. Symmetries of partial differential equations: part I *Acta Appl. Math.* **15** 161–209

[57] Chen Y-T, Lee N-C and Tu M-H 2008 A note on symmetries of WDVV equations arXiv:0808.2707

[58] Aratyn H and van de Leur J 2003 Solutions of the WDVV equations and integrable hierarchies of KP type *Commun. Math. Phys.* **239** 155–82 (arXiv:hep-th/0104092)

[59] Mokhov O I 2006 Systems of integrals in involution and the associativity equations *Russ. Math. Surv.* **61** 568–570

[60] Kunzinger M and Popovych R O 2008 Potential conservation laws *J. Math. Phys.* **49** 103506 (arXiv:0803.1156)

[61] van de Leur J 2001 Twisted $GL_n$ loop group orbit and solutions of the WDVV equations *Int. Math. Res. Not.* **2001** 551–73 (arXiv:nlin.SI/0004021)